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Along a Moishezon space I : increase, vanishing and convergence of functions.

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0. Introduction.

In this paper we clarify certain close relations among properties of the following:

(a) Polynomials on an affine algebraic variety;

(b) Meromorphic functions on a Moishezon space;

(c) Formal functions along a Moishezon subspaces;

(d) Pullbacks of function germs at a point of complex spaces;

(e) Products of function germs at a point of a complex space.

In §1 we are concerned with increase of holomorphic functions and convergence of formal functions. Sadullaev [S] has obtained an increase estimate of polynomials on an affine algebraic variety by the uniform norm on a compact domain. This can be generalized to a similar estimate for meromorphic functions on a Moishezon space. This is equivalent to the assertion that a formal function along a Moishezon subspace converges either everywhere or nowhere. On the other hand Gabrielov [G] has proved a difficult theorem on the convergence of pullbacks of formal functions. We prove that weak versions of the theorems of Sadullaev and Gabrielov are equivalent modulo algebraic geometry.

In §2 we treat vanishing orders of function germs and global meromorphic functions. In a earlier paper [I_2], the author has obtained a few basic inequalities for vanishing orders of function germs on an irreducible germ of a complex space. We prove that these inequalities are equivalent to the following statement: If $X\subset\mathbb{R}^n$ is an algebraic variety, a polynomial of a small degree can not have a high vanishing order at a prescribed point. We prove a similar theorem for a meromorphic function on a Moishezon space. This is equivalent to the assertion that, for a holomorphic function along a Moishezon subspace S, vanishing in a high order at a point ensures vanishing in a high order along entire S.

We have noticed certain analogies between problems of convergence and problems of vanishing orders in (T_2) and (I_5). This time we intentionally utilize such analogies again. Theorems (A), ..., (D), (E') in §1 correspond to (A^*), ..., (D^*), (E^*) in §2 respectively. Theorems (A), ..., (D) are mutually equivalent as well as (A^*), ..., (E^*) are. The proofs of these two series of equivalence are almost parallel. Theorem (E') is the assertion that a formal factorization of an element of a normal analytic algebra A reduces to an analytic factorization. This is also found as an analogue of (E^*) but the author did not find its relation to (A), ..., (D). Theorems (A), (D), (D^*), (E^*) are established theorems. Other theorems may be new and they are verified in §4 through the equivalence to these established theorems. Only (E') is proved independently in §5. §3 is an algebraic provision for the proofs in §4.

If the dimension of the Moishezon space S (or the analytic algebra A) is zero, (A), ..., (E'), (A^*), ..., (E^*) are trivial except the restrictions on the constants a and b in (A^*), ..., (E^*) and the restrictions are false. Therefore we assume that $\dim S \geqq 1$ and $\dim A \geqq 1$. By <u>analytic</u> <u>algebra</u> we mean the residue class algebra of a convergent power series algebra $\mathbb{C}\{x_1, \ldots x_p\}$. We express a morphism (holomorphic map) between complex spaces by a capital Greek letter and the induced homomorphism between analytic algebras or algebras of sections by the corresponding small Greek letter.

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1. Results on increase and convergence.

Let alg(S) denote the transcendental degree over $\mathbb{C}$ of the field of the meromorphic functions on a compact complex space S. It is well known that $\dim S \geqq \mathrm{alg}(S)$ (Thimm, Remmert). A compact complex space S is called a Moishezon space if all the irreducible components S_i of S_{red} satisfy $\dim S_i = \mathrm{alg}(S_i)$. We call an analytic space, an analytic algebra and so forth integral when they are reduced and irreducible. Let f be a continuous function on a topological space X and K a compact subset. We put $\|f\|_K \equiv \max_{x\in K}|f(x)|$.

THEOREM A (Sadullaev (S), (2.2)). *For any compact domain K of an integral algebraic subvariety S of $\mathbb{C}^n$, the function*

$$P_K(z) \equiv \sup\{|f(z)|^{1/d} : d \in \mathbb{N},\ f \text{ is a polynomial of degree } d \text{ such that } \|f\|_K = 1\}$$

is locally bounded on S.

(1.1) REMARK. Sadullaev's original assumption is that K is compact and not pluripolar. This original form is recovered from (A) by (S), (2.1). He has proved the converse also: if S is an integral complex subspace of $\mathbb{C}^n$, boundedness of $P_K(z)$ implies that S is algebraic.

Let D be a Cartier divisor and $L(D)$ the set of all meromorphic functions on S whose pole divisors are at most D.

THEOREM B. *Let D be a Cartier divisor on an integral Moishezon space S and $K \subset S \setminus (\mathrm{spt}\ D)$ a compact domain. Then the function*

$$P_K(z) \equiv \sup\{|f(z)|^{1/d} : d \in \mathbb{N},\ f \in L(dD) \text{ such that } \|f\|_K = 1\}$$

is locally bounded on $S \setminus (\mathrm{spt}\ D)$.

THEOREM C. *Let S be a thin connected Moishezon subspace of a reduced complex space X such that X is integral along S and $\mathcal{O}_{\hat{X}}$ the structure sheaf of the completion of X along S (cf. (BS)). If $f^{\sim} \in \Gamma(S, \mathcal{O}_{\hat{X}})$ (formal function along S) is convergent at $\xi \in S$, $f^{\sim}$ is convergent: $f^{\sim} \in \Gamma(S, \mathcal{O}_X)$.*

Let $\varphi : A \longrightarrow B$ be a homomorphism between reduced analytic algebras corresponding to a morphism $\Phi_\eta : Y_\eta \longrightarrow X_\xi$ between germs of complex spaces. We put

$$r_1(\varphi) \equiv \inf\{\text{topl-dim } \Phi(U) : U \text{ is a neighborhood of } \xi\}/2,$$
$$r_2(\varphi) \equiv \dim \hat{A}/\ker \hat{\varphi}, \qquad r_3(\varphi) \equiv \dim A/\ker \varphi,$$

where $\hat{\varphi}: \hat{A} \longrightarrow \hat{B}$ denotes the extension to maximal-ideal-adic completions. It is known that $r_1 \leqq r_2 \leqq r_3$ (cf. [I_4], §1).

THEOREM D (Gabrielov [G]). *If* $\varphi: A \longrightarrow B$ *is a homomorphism between integral analytic algebras with* $r_1 = r_3$, *then* $\hat{\varphi}(\hat{A}) \cap B = \varphi(A)$.

(1.2) REMARK. Gabrielov [G], (5.2) has proved that $r_1 = r_2$ implies $r_1 = r_2 = r_3$. (D) is its weakened corollary. See [I_4], §10 for the history and the related results. Tougeron [T_3] has simplified the proof of Gabrielov's original theorem and has improved it.

THEOREM E′. *Let* A *be a normal analytic domain. Then a formal factorization of an element of* A *reduces to an analytic one: if* $f_1, \ldots, f_p \in \hat{A}$ *and* $\prod f_i \in A$, *then there exist invertible elements* $u_1, \ldots u_p \in \hat{A}$ *such that* $u_1 f_1, \ldots, u_p f_p \in A$ *and* $\prod u_i = 1$.

By an Artin's theorem [A], a formal factorization implies an analytic factorization. But it does not imply such an factorization is equivalent to the original formal one modulo formal invertible factors.

2. Results on orders at a point and along a Moishezon subspace.

Let $\mathfrak{m}$ be the maximal ideal of a local ring A. The *order* and the *reduced order* of $f \in A$ are defined by

$$\nu(f) \equiv \nu_{A,\mathfrak{m}}(f) \equiv \sup\{p : f \in \mathfrak{m}^p\},$$
$$\bar{\nu}(f) \equiv \bar{\nu}_{A,\mathfrak{m}}(f) \equiv \lim_{p\to\infty} \nu(f^p)/p.$$

respectively. If A is the local ring $O_{X,\xi}$ (resp. the completion $\hat{O}_{X,\xi} \equiv O_{\hat{X},\xi}$ of $O_{X,\xi}$), $\nu(f)$ is denoted by $\nu_{X,\xi}(f)$ (resp. $\nu_{\hat{X},\xi}(f)$).

THEOREM A*. *Let* S *be an integral algebraic subvariety of*

$\mathbb{C}^n$. Then, for any $\xi \in S$, there exists $a \in \mathbb{R}$ such that

$$a \cdot (\deg f) \geqq \bar{\nu}_{S,\xi}(f)$$

for any polynomial $f \neq 0$ on $\mathbb{C}^n$. Such an a must satisfy $a \geqq 1$.

THEOREM B*. Let D be a Cartier divisor on an integral Moishezon space S. Then, for any $\xi \in S \setminus (\mathrm{spt}\ D)$, there exists $a \in \mathbb{R}$ such that

$$ad \geqq \bar{\nu}_{S,\xi}(f)$$

for any $f \in L(dD) \setminus \{0\}$. Such an a must satisfy $a > 0$.

Let X be a reduced complex space and I a coherent ideal sheaf. We put

$$\nu_{X,I}^{K}(f) \equiv \inf\{\sup\{p \in \mathbb{N}:\ f_x \in I_x^p\}:\ x \in K\}$$
$$\bar{\nu}_{X,I}^{K}(f) \equiv \inf\{\sup\{r/q:\ \exists\, q, r \in \mathbb{N},\ f_x^q \in I_x^r\}:\ x \in K\}$$
$$\equiv \inf\{\lim_{k\to\infty} \nu_{X,I}^{x}(f^k)/k:\ x \in K\}$$

for any subset $K \subset X$ and for any f holomorphic in a neighborhood of K. It is obvious that $\bar{\nu}_{X,I}^{K}(f) \geqq \nu_{X,I}^{K}(f)$. In the case $K = \{\xi\}$ and I is the ideal sheaf that defines $\{\xi\}$, $\nu_{X,I}^{K}(f)$ (resp. $\bar{\nu}_{X,I}^{K}(f)$) coincides with $\nu_{X,\xi}(f)$ (resp. $\bar{\nu}_{X,\xi}(f)$) defined above.

(2.1) REMARK (Lejeune-Teissier [LT], (4.1.8), (5.5)). Let $\Pi: X' \longrightarrow X$ be a proper surjective morphism between reduced complex spaces, Y a thin complex subspace of X and I a coherent ideal sheaf on X with locus Y. Suppose that X' is normal and the analytic inverse image sheaf $\Pi^*(I)$ is invertible. Then we have the following for any compact subset $K \subset Y$:
(i) There exists an arbitrarily small neighborhood U of K such that $\Pi^{-1}(Y \cap U)$ has only a finite number of irreducible components $Y_1, \ldots, Y_p$, all Y_i intersect $\Pi^{-1}(K)$ and Y_i are all irreducible along $\Pi^{-1}(K)$. (Y_i are of codimension one by Krull's Hauptidealsatz.)
(ii) If U is as above and $f \in \Gamma(U, O_X)$,

$$\bar{\nu}_{U,I}^{K}(f) = \sup\{r/q:\ \exists\, q, r \in \mathbb{N},\ (f \circ \Pi)_\eta^q \in \Pi_\eta^*(I_\eta^r)$$
$$\text{for } \forall \text{ (or } \exists)\ \eta \in \Pi^{-1}(U) \cap Y_i\ \ (i = 1, \ldots, p)\}.$$

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(2.2) REMARK (Rees [R_2], The strong valuation theorem). If $(A, \mathfrak{m})$ is a local ring whose completion is reduced, then there exists b such that

$$\nu_{A,\mathfrak{m}}(f) + b \geqq \bar{\nu}_{A,\mathfrak{m}}(f) \quad (\geqq \nu_{A,\mathfrak{m}}(f)).$$

such a b must satisfy $b \geqq 0$.
(From now on a parenthesized inequality means a trivial one.)

THEOREM C*. Let S be a thin connected Moishezon subspace of a reduced complex space X defined by a coherent ideal sheaf I. Then, for any $\xi \in S$, there exist $a \in \mathbb{R}$ such that

$$a \cdot \bar{\nu}_{X,I}^{S}(f^{\sim}) \geqq \bar{\nu}_{X,\xi}(f^{\sim}) \quad (\geqq \bar{\nu}_{X,I}^{S}(f^{\sim}))$$

for any $f^{\sim} \in \Gamma(S, O_X)$. Such an a must satsfy $a > 1$.

(2.3) REMARK. Let X be a normal complex space and S and T its hypersurfaces defined by invertible ideal sheaves I and J respectively. Suppose that S is an irreducible Moishezon space, T intersects S and T is irreducible along S. If f is a representative of a fixed $f^{\sim} \in \Gamma(S, O_X)$ defined on a neighborhood U of S such that $T \cap U$ is irreducible, then $\nu_{U,I}^{x}(f)$ (resp. $\nu_{U,J}^{y}(f)$) is independent of (U, f) and $x \in S$ (resp. $y \in U \cap T$) by Riemann's second removable singularity theorem. This value $v_S(f^{\sim})$ (resp. $v_T(f^{\sim})$) defines a valuation on $\Gamma(S, O_X)$. Then (C^*) implies the existence of $c > 0$ with

$$c \cdot v_S(f^{\sim}) \geqq v_T(f^{\sim})$$

for any $f^{\sim} \in \Gamma(S, O_X)$ (cf. (2.1), (ii)). This situation arises in the case of a resolution of a non-isolated singularity.

THEOREM D* (Izumi [I_2]). If $\varphi: A \longrightarrow B$ is an injective homomorphism between integral analytic algebras with $r_1 = r_3$, then there exist $a, b \in \mathbb{R}$ such that

$$a \cdot \nu(f) + b \geqq \nu(\varphi(f)) \quad (\geqq \nu(f))$$

for any $f \in \hat{A}$. Such a and b must satisfy $a \geqq 1$, $b \geqq 0$.

(2.4) REMARK. The case A is regular is proved by Tougeron [T_1], (II, 1.3). It is known that the converse of (D^*) is also true: the inequality for suitable $a, b \in \mathbb{R}$ implies $r_1 = r_3$ (see

(I_3)).

THEOREM E* (Izumi (I_2), (3.4); cf. (I_3), (1.6)). <u>If</u> A <u>is an integral analytic algebra, there exists</u> $a \in \mathbb{R}$ <u>such that</u>

$$a \cdot \{\bar{\nu}(f) + \bar{\nu}(g)\} \geqq \bar{\nu}(fg) \quad (\geqq \bar{\nu}(f) + \bar{\nu}(g)).$$

<u>Such an</u> a <u>must satisfy</u> $a \geqq 1$.

3. A Moishezon subspace is an image of a negative subspace

The lemmas of this section will reveal the reason why a property at a point of a connected Moishezon subspace $S \subset X$ influences properties at other points as in (C) and (C*). In short, the reason is that S can be transformed into a point by birational transformations of X.

(3. 1) LEMMA. Let S be a Moishezon subspace of a reduced complex space X_0. Then there exists a thin analytic subspace Z of a neighborhood $X \subset X_0$ of S such that the blowing up $\Phi : X' \longrightarrow X_0$ with center Z satisfies the following:
(i) X' is smooth.
(ii) $\Phi^{-1}(S)$ is defined by an invertible sheaf of ideals.
(iii) All components T_α of the total transform $\Phi^{-1}(S)_{red}$ are smooth projective varieties.

PROOF of (3. 1). First note the following facts. The Hironaka resolution of singularity (AHV) is a finite composition of blowings up on a relatively compact subset. A finite composition of blowings up is a blowing up again on a relatively compact subset (cf. (HR), (5)). For a blowing up, subvarieties of the inverse image of a Moishezon space (resp. a projective variety) are also Moishezon (resp. projective).

We prove by induction on $s = \dim S$. The case $s = 0$ is obvious. So we assume the case $s-1$ ($s \geqq 1$). By Moishezon's theory (M), there exists a thin subspace $T_1 \subset S$ such that the strict transform of S is projective for the blowing up $\Pi_1 : X_1 \longrightarrow X_0$ with center T_1. By the inductive hypothesis, there exists thin subspace T_2 of a neighborhood $U \subset X_0$ of S such that all the components of the total transform of T_1 with respect to the blowing up $\Pi_2 : X_2 \longrightarrow U$ with center T_2 are projective. Let $T \subset U$ be the subspace determined by the product of the ideal sheaves of $T_1|U$ and T_2. By (Hi), (2. 10), the blowing up Π with center T dominates $\Pi_1|\Pi_1^{-1}(U)$ and Π_2 through blowings up. Then all the components of the total transform S' of S with respect to Π are projective. We have only to compose Π and the Hironaka resolution and choose a relatively compact neiborhood $X \subset U$ of S. ▯

(3. 2) LEMMA (cf. (M), II, Th. 5). <u>Let</u> X' <u>be a smooth complex manifold and</u> T <u>its compact smooth hypersurface. Assume that</u> T <u>is projective and that</u> H <u>is its smooth hyperplane section. Let</u> $\Psi : X'' \longrightarrow X'$ <u>be the blowing up with center</u> nH $(n \in \mathbb{N})$ <u>and</u> T' <u>the strict transform of</u> T. <u>If</u> n <u>is sufficiently large,</u> T' <u>has a line bundle neighborhood (not the total one) in</u> X'' <u>and</u> T' <u>is exceptional (in the sense of</u> (Gr)).

(3. 3) REMARK. Generic hyperplane sections of T are smooth by Bertini's theorem.

PROOF of (3. 2). Let $E = \Psi^{-1}(H) \cap T'$ be the exceptional divisor in T'. By the calculation of (F), Appendix B, (6. 10), we have

$$N_{T' | X''} \cong (\Psi | T')^* (N_{T | X'}) \otimes (-n(E)),$$

where $N_{T' | X''}$ (resp. $N_{T | X'}$) denotes the normal bundle of T' in X'' (resp. T in X') and (E) the line bundle associated to E. Since nH is a divisor, $\Psi | T'$ is an isomorphism. By a calculation of curvature tensor (cf. (Gf), (2. 10)) we see that, if n is sufficiently large, $(\Psi | T')^* (N_{T | X'}) \otimes (-n(E))$ is Griffiths negative. Then $N_{T' | X''}$ is weakly negative by (Gf), (3. 4) and T is exceptional by (Gr), Satz 8. □

4. Proofs of the equivalence.

(4.1) PROOF of the restrictions on the constants a, b in (A^*) $\sim (E^*)$.

The proof of these restrictions are independent of the mutual implications among the theorems. When we deduce (C^*) and (E^*) from (B^*) or (D^*), there remains an additive constant "b" on the left sides of the inequality. But we can delete them using the homogeneity of the superlined orders: $\bar{\nu}(f^p) = p\cdot\bar{\nu}(f)$ $(p\in\mathbb{N})$ etc. (cf. (R_1)). In (A^*), $a\geqq 1$ follows from the following observation. Let f be a generic linear function which vanishes at ξ. Then

$$ap = a(\deg f^p) \geqq \bar{\nu}_{S,\xi}(f^p) \geqq p \quad (p\in\mathbb{N})$$

implies that $a\geqq 1$. Since we have assumed that $\dim S \geqq 1$, $\bar{\nu}_{S,\xi}(f_\xi)$ is unbounded. Then a must be positive in (B^*). The restriction $a\geqq 1$ in (D^*) and (E^*) follows from the parenthesized trivial parts of the inequalities. The proof of $a>1$ in (C^*) is not trivial. It is a consequence of the last inequality of (4.5) below and unboundedness of $\bar{\nu}_{S,\xi}(f)$. Positivity of b in (D^*) follows if we put $f=1$. □

(4.2) PROOF of $(A)\Longrightarrow(B)_{Proj}$, $(A^*)\Longrightarrow(B^*)_{Proj}$.

Here $_{Proj}$ indicate the case S is projective. We may assume that D is effective. Let $S\subset P^n$ be an embedding into a projective space with a homogeneous coordinate system X_0, $\dots, X_n$. Then there exists a homogeneous polynomial g of degree p such that

$$D = S\cap\{X=(X_0,\dots,X_n) : g(X)=0\} \text{ (scheme theoretically).}$$

If we define an embedding $\Phi : S\setminus(\mathrm{spt}\, D)\longrightarrow \mathbb{C}^{n^2}$ by

$$\Phi(X) = (X_0^p/g(X),\ X_0^{p-1}X_1/g(X), \dots,\ X_n^p/g(X)) \quad (p=\deg g),$$

$f\in L(dD)$ is expressed as $f = h\circ\Phi$ by a homogeneous polynomial h of degree at most d on $\mathbb{C}^{n^2}$. Then (A) (resp. (A^*)) for $\Phi(S)$ implies (B) (resp. (B^*)) for S. □

(4.3) PROOF of $(B)\Longrightarrow(A)$, $(B^*)\Longrightarrow(A^*)$.

Let $D\subset P^n$ be the divisor defined by $X_0=0$ and identify

$\mathbb{C}^n$ with $P^n \setminus (\operatorname{spt} D)$. □

(4.4) PROOF of $(B)_{proj} \Longrightarrow (C)$, $(B^*)_{proj} \Longrightarrow (C^*)$.

Let $\varphi: A \longrightarrow B$ be an injective finite morphism between local rings. It is easy to see that $\hat{\varphi}^{-1}(B) = A$ (a special case of (D), cf. (I_1), (2.2), (2.4)) and $a\nu(f) + b \geqq \nu(\varphi(f))$ for some $a, b \in \mathbb{R}$ (a special case of (D^*), cf. (I_1), (4.1)). Hence we may assume that X is normal. We may also assume that S is irreducible. Shrink X and take a blowing up $\Phi: X' \longrightarrow X$ that satisfies the conditions in (3.1). Let T_α be a component of $\Phi^{-1}(S)$ that contains a point $\eta \in \Phi^{-1}(\xi)$. Applying (3.2) to this T_α we have a blowing up $\Psi: X'' \longrightarrow X'$ such that the strict transform T_α' of T_α has a line bundle neighborhood. Let $\Pi: N \longrightarrow T_\alpha'$ be the normal bundle of $T_\alpha' \subset X''$ and Z its zero section. Then we have a natural isomorphism between the completions $\hat{N}$ of N along Z and $\hat{X}''$ along T_α'. We identify them. By Serre's GAGA (Sr) N has an algebraic trivialization:

$$T_\alpha' = U_0 \cup \ldots \cup U_p \quad (U_i: \text{ affine Zariski open}),$$
$$\Pi^{-1}(U_i) \cong U_i \times \mathbb{C} \ni (x_i, t_i),$$
$$g_{ij}: \text{ regular functions on } U_i \text{ with } t_i = g_{ij} t_j.$$

$$\begin{array}{ccccc}
X & \xleftarrow{\ \Phi\ } & X' & \xleftarrow{\ \Psi\ } & X'' \\
\cup & & \cup & & \\
S & \longleftarrow & T & & \cup \\
\Psi & & \cup & & \\
\xi & & T_\alpha & \longleftarrow & T_\alpha' \\
& \nwarrow & \Psi & & \Psi \\
& & \eta & \longleftarrow & \zeta
\end{array}$$

We may assume that $\Psi^{-1}(\eta) \cap U_0 \neq \phi$. Take a point $\zeta \in \Psi^{-1}(\eta) \cap U_0$. If $f^\sim \in \Gamma(S, O_{\hat{X}})$, we have a unique expression

$$f^\sim \circ \hat{\Phi} \circ \Psi(x_i, t_i) = \sum_{d \geq 0} f_{i;d}(x_i) t_i^d \quad (f_{i;d} \in \Gamma(U_i, O_X))$$

on U_i. Hence

$$f_{i;d}/f_{j;d} = t_j^d / t_i^d = g_{ji}^d \qquad \text{on } U_i \cup U_j.$$

There exists a Cartier divisor D on T_α' which corresponds to the dual of N (see e.g. (H), (6.14.1)) such that $\zeta \notin (\operatorname{spt} D)$.

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Refining the covering $\{U_i\}$ if necessary, we have a system of local equations $\{\varphi_i\}$ of D ($\varphi_i \in \Gamma(U_i, \mathcal{K}^*)$, $g_{ij} = \varphi_j/\varphi_i$, $\mathcal{K}^*$ is the sheaf of the invertible rational functions on S). Putting $h = f_{i;d}/\varphi_i^d$ on U_i, we have $h^\sim \in L(dD)$.

Suppose that $f^\sim$ is convergent at ξ. Then there exist a compact neighborhood $K \subset U_0$ of ξ and positive numbers M and A such that

$$(*) \qquad \|f_{0;d}\|_K \leqq MA^d.$$

We claim for any compact $G \subset U_i \setminus (\mathrm{spt}\ D)$ there exist positive numbers M' and A' such that

$$(**) \qquad \|f_{i;d}\|_G \leqq M' A'^d.$$

Since U_i is Zariski open, moving ξ a little and shrinking K, we may assume that $K \subset U_i \setminus (\mathrm{spt}\ D)$. (*) implies

$$\|(f_{i;d}/\varphi_i^d)/M(AB)^d\|_K \leqq 1 \qquad (B \equiv \|1/\varphi_0\|_K > 0).$$

By (B) we have

$$\|(f_{i;d}/\varphi_i^d)/M(AB)^d\|_G \leqq M'' A''^d,$$

so that

$$\|f_{i;d}\|_G \leqq MM''(AA''BC)^d \qquad (C \equiv \|\varphi_i\|_G > 0).$$

This proves (**). Hence $f^\sim \circ \hat{\Phi} \circ \hat{\Psi}$ is convergent on $T_\alpha' \setminus (\mathrm{spt}\ D)$. Since Ψ is an isomorphism outside the center nH of Ψ, $f^\sim \circ \hat{\Phi}$ is convergent on $|T_\alpha| \setminus (|H| \cup \Psi(\mathrm{spt}\ D))$. Moving D and H, we see that $f^\sim \circ \hat{\Phi}$ is convergent on T_α. If $T_\alpha \cap T_\beta \neq \phi$, we can similarly see the convergence on T_β starting from any point $\xi_\beta \in T_\alpha \cap T_\beta$. Repeating this we know that $f^\sim \circ \hat{\Phi}$ is convergent on $\Phi^{-1}(S)$, since it is connected. Hence $f^\sim$ is convergent by normality. This completes the proof of $(B)_{proj} \Longrightarrow (C)$.

To prove $(B^*)_{proj} \Longrightarrow (C^*)$ we claim that there exists $a \in \mathbb{R}$ such that

$$(\dagger) \qquad a \cdot \nu^{T_\alpha}_{X',J_\alpha}(f^\sim \circ \Phi) \geqq \bar{\nu}_{X,\eta}(f^\sim),$$

where J_α denotes the ideal sheaf that defines T_α. Consider the expression

$$f^{\sim}\circ\Phi\circ\Psi = \sum_{d\geqq 0} f_{i;d}(x_i)t_i^d$$

on $U_i \subset T_\alpha'$. If we put $k^{\sim} \equiv f_{i;d}/\varphi_i^d$ on U_i, $k^{\sim} \in L(dD)$. If $U_i \setminus (\mathrm{spt}\ D) \ni \zeta$ and if $f_{i;d} \neq 0$, there exists $a' \in \mathbb{R}$ by (B^*) such that

$$a'd \geqq \bar{\nu}_{T'_\alpha,\zeta}(k^{\sim}) = \bar{\nu}_{T'_\alpha,\zeta}(f_{i;d}) \geqq \bar{\nu}_{X',\zeta}(f^{\sim}\circ\Phi\circ\Psi) - d \geqq \bar{\nu}_{X,\xi}(f^{\sim}) - d.$$

Then

$$(a'+1)d \geqq \bar{\nu}_{X,\xi}(f^{\sim}).$$

Thus we have proved that $(a'+1)d < \bar{\nu}_{X,\xi}(f^{\sim})$ implies $f_{i;d} = 0$. Since Ψ induces isomorphisms at general points of T_α', we have

$$(a'+1)\bar{\nu}_{X',J_\alpha}^{y}(f^{\sim}\circ\Phi) \geqq \bar{\nu}_{X,\xi}(f^{\sim}),$$

for general $y \in T_\alpha$. By the smoothness (3.1), (i), (iii), this holds for all $y \in T_\alpha$ and (†) follows. Such estimates are also possible for all components T_β as in the case of $(B) \Longrightarrow (C)$. Then (C^*) follows from (2.1). □

(4.5) PROOF of $(C) \Longrightarrow (B)$, $(C^*) \Longrightarrow (B^*)$.

Let $\Pi: L \longrightarrow S$ be the line bundle associated to D and $\{(U_i, \varphi_i)\}$ its algebraic trivialization (as in (4.4)).

To prove $(C) \Longrightarrow (B)$, suppose that $K \subset S \setminus (\mathrm{spt}\ D)$ is a compact domain and $P_K(z)$ is not bounded on a compact set $G \subset S \setminus (\mathrm{spt}\ D)$. We may assume that G and K are contained in U_0. Then there exist $d_\nu \in \mathbb{N}$, $f^\nu \in L(d_\nu D)$ such that $\|f^\nu\|_K = 1$, $\|f^\nu\|_G \geqq \nu^{d_\nu}$. Since the cohomology groups of a projective variety are finite dimensional and since any two norms of a finite dimensional $\mathbb{C}$-vector spaces are equivalent, $d_\nu \longrightarrow \infty$. We may assume that $d_1 < d_2 < \ldots$. If we put

$$f^{\sim} \equiv \sum_{\nu>0} f^\nu(x_0)\varphi_0(x_0)^{d_\nu} t_0^{d_\nu} \quad \text{on } U_0,$$

$f^{\sim} \in \Gamma(S, O_{\hat{L}})$, where $\hat{L}$ denotes the completion of L along its zero section. Since φ_0 and $1/\varphi_0$ are bounded on G and K, $f^{\sim}$ is convergent on K and divergent at some point of G, a contradiction to (C).

To prove $(C^*) \Longrightarrow (B^*)$, suppose that $f \in L(dD) \setminus \{0\}$. Then, if we put $f^{\sim} \equiv f\varphi_i^d t_i^d$ on U_i, $f^{\sim} \in \Gamma(S, O_L)$ (cf. (4.4)) so that

$$
\begin{aligned}
ad = a\bar{\nu}_{X,1}^{s}(f^{\sim}) &\geqq \bar{\nu}_{X,\xi}(f^{\sim}) \qquad \text{(by } (C^*))\\
&\geqq \bar{\nu}_{S,\xi}(f) + d.
\end{aligned}
$$

Hence $(a-1)d \geqq \bar{\nu}_{S,\xi}(f)$. This proves (B^*). □

(4.6) PROOF of $(C) \Longrightarrow (D)$, $(C^*) \Longrightarrow (D^*)$.

Quite similar to $[I_1]$, (6.2), (6.3). (There we have used resolution of singularity and the regular cases of (D) (Moussu-Tougeron, Malgrange, Eakin-Harris) and (D^*) (Tougeron cf. (2.4)). We have used (2.1) also to prove $(C^*) \Longrightarrow (D^*)$.) □

(4.7) PROOF of $(D) \Longrightarrow (C)$, $(D^*) \Longrightarrow (C^*)$.

We may assume that S is irreducible. By (3.1) and (3.2) we have the following diagram (shrinking X).

$$
\begin{tikzcd}
X & X' \arrow[l, "\Phi"'] & X'' \arrow[l, "\Psi"'] \arrow[r, "\Pi"] & Y \\
\cup & \cup & & \\
S & T \arrow[l] & \cup & \rotatebox{90}{$\in$} \\
\rotatebox{90}{$\in$} & \cup & & \\
\xi & T_\alpha & T_\alpha' \arrow[l] \arrow[r] & \zeta' \\
 & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
 & \eta \arrow[uul] & \zeta \arrow[l] &
\end{tikzcd}
$$

Here T_α' is exceptional and Π is the associated contraction: $\Pi_*(O_{X''}) \cong O_Y$. For any $f \in \Gamma(S, O_{\hat{X}})$ there exists $g \in O_{\hat{Y},\zeta'}$ such that $f \circ \hat{\Phi} \circ \hat{\Psi} = g \circ \hat{\Pi}$ by the comparison theorem applied to Π (cf. [BS], Ⅶ, §4). Then we have the following implications:

$f^{\sim}_{\xi}$ is convergent

$\Longrightarrow$ $(f^{\sim} \circ \hat{\Phi} \circ \hat{\Psi})_{\zeta} = (g \circ \hat{\Pi})_{\zeta}$ is convergent

$\Longrightarrow$ $g_{\zeta'}$ is convergent (by (D))

$\Longrightarrow$ $g \circ \hat{\Pi} = f^{\sim} \circ \hat{\Phi} \circ \hat{\Psi}$ is convergent on T_α'

$\Longrightarrow$ $f^{\sim} \circ \hat{\Phi}$ is convergent on T_α (by (D))

$\Longrightarrow$ $f^{\sim} \circ \hat{\Phi}$ is convergent on T

(Apply the above arguments to the neighboring T_β. Repeating this, we see the convergence on the whole T, since it is connected.)

$\Longrightarrow$ $f^{\sim} \in \Gamma(S, O_X)$ (by (D)).

This proves $(D) \Longrightarrow (C)$.

To prove $(D^*) \Longrightarrow (C^*)$, we have only to replace "be convergent" (resp. (D)) by "have a high order of vanishing" (resp. (D^*)) in the above (cf. (4.1)). □

(4.8) PROOF of $(D^*) \Longleftrightarrow (E^*)$.

See Proof of (1_1), (1.2) (cf. (4.1)). (There we have used the resolution of singularity for ($\Longrightarrow$) and Tougeron's idea in (T_2) for ($\Longleftarrow$).) □

5. Proof of (E').

(5.1) LEMMA. If A is a regular analytic algebra, (E') is true.

PROOF. We have only to prove the case $p=2$. So suppose that $f, g \in \hat{A}$ and $fg \in A$. Let

$$f = h_1^{p_1} \ldots h_r^{p_r}, \qquad g = h_1^{q_1} \ldots h_r^{q_r}$$
$$(h_i \hat{A} \neq h_j \hat{A} \text{ if } i \neq j;\ p_i \geqq 0,\ q_i \geqq 0,\ p_i + q_i > 0)$$

be the factorization into prime elements in $\hat{A}$. Suppose that $i \neq j$ and $h_i \in h_j \hat{A} + \mathfrak{m}^t$ for any $t \in \mathbb{N}$. Then $h_i \in h_j \hat{A}$ by Krull's intersection theorem. Hence we have $h_i \hat{A} = h_j \hat{A}$, a contradiction. This proves that there exists $q \in \mathbb{N}$ such that $h_i \notin h_j \hat{A} + \mathfrak{m}^q$ if $i \neq j$. By Artin's theorem (A) on analytic equations, there exist $k_1, \ldots, k_r \in A$ such that

$$fg = k_1^{p_1+q_1} \ldots k_r^{p_r+q_r},$$
$$k_i - h_i \in \mathfrak{m}^q.$$

By the latter, k_i is not equivalent to h_j modulo an invertible factor ($i \neq j$). Since $\hat{A}$ is factorial, there exist invertible elements $w_1, \ldots, w_r$ such that $k_i = w_i h_i$. Putting

$$u = w_1^{p_1} \ldots w_r^{p_r}, \qquad v = w_1^{q_1} \ldots w_r^{q_r}$$

we have $uf \in A$, $vg \in A$ and $uv = 1$. Thus $(uf)(vg)$ realizes an analytic factorization of fg. □

(5.2) PROOF of (E').

We have only to prove the case $p=2$. So suppose that $f, g \in \hat{A}$ and $fg \in A$. Take a normal complex space X such that A=

$O_{X,\xi}$. Let $\Pi: Y \longrightarrow X$ be a Hironaka resolution and $\hat{X}$ (resp. $\hat{Y}$) the completion of X at ξ (resp. Y along $S \equiv (\Pi^{-1}(\xi))_{red}$). The morphism Π induces a formal morphism $\hat{\Pi}: \hat{Y} \longrightarrow \hat{X}$, and homomorphisms $\pi: A \longrightarrow \Gamma(S, O_Y)$ and $\hat{\pi}: \hat{A} \longrightarrow \Gamma(S, O_{\hat{Y}})$. The section $\pi(fg) \in \Gamma(S, O_Y)$ has a formal factorization $\pi(fg) = \hat{\pi}(f)\hat{\pi}(g)$. By (5.1) there exist a covering $\cup U_i = S$ and invertible sections $w_i \in \Gamma(U_i, O_{\hat{Y}}^*)$ such that $f_i \equiv w_i \pi(\hat{f}) \in \Gamma(U_i, O_Y)$, $w_i^{-1}\hat{\pi}(g) \in \Gamma(U_i, O_Y)$. Since $f_i/f_j \equiv w_i/w_j \in \Gamma(U_i \cap U_j, O_Y^*)$, they determine an element $L \in \mathrm{Pic}(Y) \equiv H^1(S, O_Y^*)$ the image of which vanishes in $\mathrm{Pic}(\hat{Y})$. Bingener deduced the injectivity of the canonical morphism $\mathrm{Pic}(Y|S) \longrightarrow \mathrm{Pic}(\hat{Y})$ from the injectivity of $R^n\Pi_*(O_Y)_\xi \longrightarrow R^n\hat{\Pi}_*(O_{\hat{Y}})_\xi$ (see [B], Proof of (3.1)). Thus we have $L = 0$ and hence there exist a refinement $\{V_j\}$ of $\{U_i\}$ with refining map ρ of indexes and $v_j \in \Gamma(V_j, O_Y^*)$ such that $f_{\rho(k)}/f_{\rho(j)} \equiv v_k/v_j$. Putting $u^\sim \equiv w_{\rho(j)}/v_j$ on V_j, we have $u^\sim \in \Gamma(S, O_{\hat{Y}}^*)$ and $u^\sim\hat{\pi}(f) \in \Gamma(S, O_Y)$. By normality Π satisfies $\Pi_*(O_Y) = O_X$ and, by the comparison theorem, $\hat{A} \cong \Gamma(S, O_{\hat{Y}})$ (see [BS], VI, (4.5), (4.7)). Then there exists $h \in A$ and $u \in \hat{A}^*$ such that $u^\sim = \hat{\pi}(u)$ and $u^\sim\hat{\pi}(f) = \pi(h)$. This proves $uf = h \in A$. Obviously u is invertible. Similarly we have $u^{-1}g \in A$. Thus we have an analytic factorization $fg = (uf)(u^{-1}g)$. □

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